Ex 1

List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$. **b)** $a + b = 4$.
c) $a > b$. **d)** $a \mid b$.
e) $\gcd(a, b) = 1$. **f)** $\text{lcm}(a, b) = 2$.

Ex 2

a) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.
b) Display this relation graphically, as was done in Example 4.
c) Display this relation in tabular form, as was done in Example 4.

Solution 1

(a) $R = \{(0,0),(1,1),(2,2),(3,3)\}$
(b) $R = \{(1,3),(2,2),(3,1),(4,0)\}$
(c) $R = \{(1,0),(2,0),(3,0),(4,0),(2,1),(3,1),(4,1),(3,2),(4,2),(4,3)\}$
(d) $R = \{(1,0),(2,0),(3,0),(4,0),(1,1),(1,2),(2,3),(1,3),(3,3)\}$
(e) $R = \{(1,0),(0,1),(1,1),(1,2),(1,3),(2,1),(3,1),(4,1),(2,3),(3,2),(4,3)\}$
(f) $R = \{(1,2),(2,1),(2,2)\}$

Given:

$$A = \{0,1,2,3,4\}$$

$$B = \{0,1,2,3\}$$

(a)

$$R = \{(a,b)|a = b\}$$

We note that the elements common by A and B are $0,1,2$ and 3.

R then contains the points with $a = b = 0$, $a = b = 1$, $a = b = 2$ and $a = b = 3$

$$R = \{(0,0),(1,1),(2,2),(3,3)\}$$

(b)

$$R = \{(a,b)|a+b=4\}$$

We note that the sum of the element of A and the element of B needs to be equal to 4:

$$a = 1, b = 3$$

$$a = 2, b = 2$$

$$a = 3, b = 1$$

$$a = 4, b = 0$$

R then contains the points:

$$R = \{(1,3),(2,2),(3,1),(4,0)\}$$

(c)

$$R = \{(a,b)|a > b\}$$

We note that the element of A needs to be larger than the element of B:

$$a = 1, b = 0$$

$$a = 2, b = 0$$

$$a = 3, b = 0$$

$$a = 4, b = 0$$

$$a = 2, b = 1$$

$$a = 3, b = 1$$

$$a = 4, b = 1$$

$$a = 3, b = 2$$

$$a = 4, b = 2$$

$$a = 4, b = 3$$

R then contains the points:

$$R = \{(1,0),(2,0),(3,0),(4,0),(2,1),(3,1),(4,1),(3,2),(4,2),(4,3)\}$$

(d)

$$R = \{(a,b)|a|b\}$$

We note that the element of A needs to be a divisor of the element of B.

All nonnegative integers are divisors of 0 (as 0 divided by any nonnegative integer is 0

$$a = 1, b = 0$$

$$a = 2, b = 0$$

$$a = 3, b = 0$$

$$a = 4, b = 0$$

The only divisor of 1 is 1 itself.

$$a = 1, b = 1$$

The divisors of 2 are 1 and 2.

$$a = 1, b = 2$$

$$a = 2, b = 2$$

The divisors of 3 are 1 and 3

$$a = 1, b = 3$$

$$a = 3, b = 3$$

R then contains the points:

$$R = \{(1,0),(2,0),(3,0),(4,0),(1,1),(1,2),(2,2),(1,3),(3,3)\}$$

(e)

$$R = \{(a, b) | gcd(a, b) = 1\}$$

We note that the greatest common divisor of the element of A and the element of B needs to be equal to 1. In other words, 1 can be the only common divisor of the two elements.

Since 0 is divisible by any nonnegative integer and since 1 is only divisible by 1, the greatest common divisor of 0 and 1 is 0.

$$a = 1, b = 0$$

$$a = 0, b = 1$$

Since 1 is only divisible by 1, the greatest common divisor between 1 and any positive integer is always 1.

$$a = 1, b = 1$$

$$a = 1, b = 2$$

$$a = 1, b = 3$$

$$a = 2, b = 1$$

$$a = 3, b = 1$$

$$a = 4, b = 1$$

The greatest common divisor of 2 and 3 is 1.

$$a = 2, b = 3$$

$$a = 3, b = 2$$

The greatest common divisor of 2 and 4 is 2.
The greatest common divisor of 3 and 4 is 1.

$$a = 4, b = 3$$

R then contains the points:

$$R = \{(1,0),(0,1),(1,1),(1,2),(1,3),(2,1),(3,1),(4,1),(2,3),(3,2),(4,3)\}$$

(f)

$$R = \{(a, b) | lcm(a, b) = 2\}$$

We note that the least common multiple of the element of A and the element of B needs to be equal to 2.

Since the least common multiple is 2, neither element can be larger than 2.

Since any multiple of 0 is always 0 itself, the least common multiple of 0 and any other integer does not exist.

If both elements are 1, then the least common multiple is 1.

The only remaining possibility is then that at least one of the two elements is a 2 and the other element is a 1 or 2. In each of these cases, the least common multiple is 2.

$$a = 1, b = 2$$

$$a = 2, b = 1$$

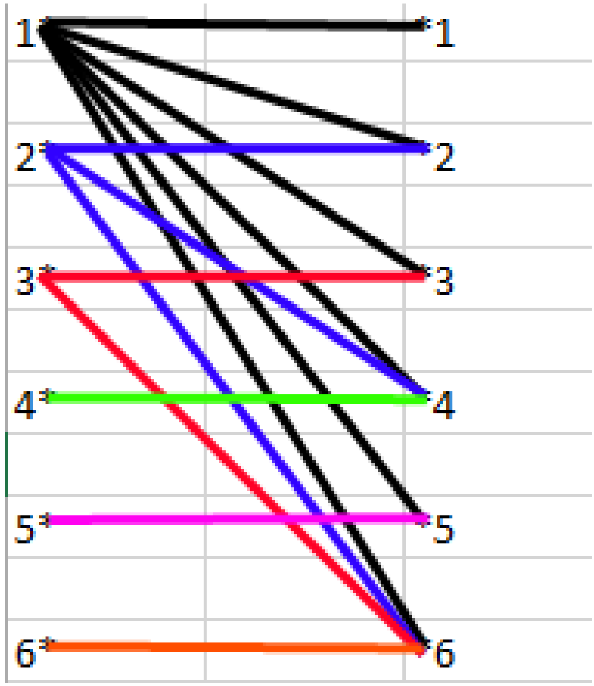
$$a = 2, b = 2$$

R then contains the points:

$$R = \{(1, 2), (2, 1), (2, 2)\}$$

Solution 2

A. R = {(1,1), (1,2), (1,3), (1,4), (1,5), (1.6),
(2,2,), (2,4), (2,6),
(3,3), (3,6),
(4,4),
(5,5),
(6,6)}



R	1	2	3	4	5	6
1	x	x	x	x	x	x
2		x		x		x
3			x			x
4				x		
5					x	
6						x

Ex 3

Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x \neq y$.

b) $xy \geq 1$.

c) $x = y + 1$ or $x = y - 1$.

d) $x \equiv y \pmod{7}$.

e) x is a multiple of y.

f) x and y are both negative or both nonnegative.

g) $x = y^2$.

h) $x \geq y^2$.

Solution 3

DEFINITIONS

A relation R on a set A is **reflexive** if $(a,a) \in R$ for every element $a \in A$.
A relation R on a set A is **symmtric** if $(b,a) \in R$ whenever $(a,b) \in R$
A relation R on a set A is **antisymmtric** if $(b,a) \in R$ and $(a,b) \in R$ implies $a = b$
A relation R on a set A is **transitive** if $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$

SOLUTION

$$A = \text{Set of all integers}$$

(a)

$$R = \{(x,y) | x \neq y\}$$

The relation R is **not reflexive**, because $x \neq x$ is never true.

The relation R is **symmetric**, because if $x \neq y$, then $y \neq x$.

The relation R is **not antisymmetric**, because $1 \neq 2$ and $2 \neq 1$, while 1 and 2 are not the same.

The relation R is **not transitive**, because $1 \neq 2$ and $2 \neq 1$, while $1 = 1$.

(b)

$$R = \{(x, y) | xy \geq 1\}$$

The relation R is **not reflexive**, because when $x = 0$: $xx = x^2 = 0 < 1$ and thus $(0, 0) \notin R$.

The relation R is **symmetric**, because if $xy \geq 1$, then $yx = xy \geq 1$ (commutative property of multiplication).

The relation R is **not antisymmetric**, because $1 \cdot 2 \geq 1$ and $2 \cdot 1 \geq 1$, while $1 \neq 2$.

The relation R is **transitive**, because if $xy \geq 1$ and if $yz \geq 1$, then x and y are nonzero integers of the same sign, and y and z are nonzero integers of the same sign. Then x and z are nonzero integers of the same sign and thus $xz \geq 1$.

(c)

$$R = \{(x, y) | x = y + 1 \text{ or } x = y - 1\}$$

The relation R is **not reflexive**, because $x \neq x + 1$ and $x \neq x - 1$ for any integer x.

The relation R is **symmetric**, because if $x = y + 1$, then $y = x - 1$ and if $x = y - 1$, then $y = x + 1$.

The relation R is **not antisymmetric**, because if $x = 1$ and $y = 2$, then $(x, y) \in R$ and $(y, x) \in R$, while $x \neq y$

The relation R is **not transitive**, because if $x = 1$, $y = 2$ and $z = 3$, then $(x, y) \in R$ and $(y, z) \in R$, while $(x, z) \notin R$.

(d)

$$R = \{(x,y)|x \equiv y \pmod 7\}$$

The relation R is **reflexive**, because $x \equiv x \pmod 7$ is true for any integer x.

The relation R is **symmetric**, because if $x \equiv y \pmod 7$, then $y \equiv x \pmod 7$

The relation R is **not antisymmetric**, because if $x = 0$ and $y = 7$, then $x \equiv y \pmod 7$ and $y \equiv x \pmod 7$, while $x \neq y$.

The relation R is **transitive**, because if $x \equiv y \pmod 7$ and $x \equiv z \pmod 7$, then $x \equiv z \pmod 7$.

(e)

$$R = \{(x, y) | x \text{ is a multiple of } y\}$$

The relation R is **reflexive**, because x is always a multiple of itself.

The relation R is **not symmetric**, because 2 is a multiple of 1, while 1 is not a multiple of 2.

The relation R is **not antisymmetric**, because -1 is a multiple of 1 and 1 is a multiple of -1, while $1 \neq -1$.

The relation R is **transitive**, because if x is a multiple of y and if y is a multiple of z, then x also has to be a multiple of z.

(g)

$$R = \{(x, y)|x = y^2\}$$

The relation R is **not reflexive**, because (for example) $(2, 2)$ is not in R (since $2 \neq 2^2 = 4$).

The relation R is **not symmetric**, because if $x = 4$ and $y = 2$, then $4 = 2^2$ and $2 \neq 4^2$

The relation R is **antisymmetric**, because if $x = y^2$ and $y = x^2$, then $x = y^2 = (x^2)^2 = x^4$, which implies $x = y = 0$ or $x = y = 1$.

The relation R is **not transitive**, because if $x = 16$, $y = 4$ and $z = 2$, then $x = y^2$ and $y = z^2$, while $x \neq z^2$

(h)

$$R = \{(x, y) | x \geq y^2\}$$

The relation R is **not reflexive**, because (for example) (2,2) is not in R (since $2 < 2^2$).

The relation R is **not symmetric**, because if $x = 2$ and $y = 1$, then $2 \geq 1^2$ and $1 < 2^2$.

The relation R is **antisymmetric**, because if $x \geq y^2$ and if $y \geq x^2$, then $x \geq y^2 \geq (x^2)^2 = x^4$ and $x \geq x^4$ is only true if $x = 0$ or $x = 1$. When $x = 0$, then $y = 0$ also. When $x = 1$, then $1 \geq y^2 \geq 1$, thus $y^2 = 1$ and thus also $y = 1$ (as $y \geq x^2$).

The relation R is **transitive**, because if $x \geq y^2$ and if $y \geq z^2$, then $x \geq y^2 \geq y \geq z^2$.